5 Harmonic Functions

5.1 Mean Value and Maximum

We coordinatize the complex plane by coordinates $(x, y) \in \mathbb{R}^2$ with $z = x + \mathrm{i}y \in \mathbb{C}$. The Laplace operator on the complex plane is then given by

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Definition 5.1. Let $U \subseteq \mathbb{C}$ be an open set and $f : U \to \mathbb{R}$ be twice continuously partially differentiable. Then, f is called *harmonic* iff it satisfies the *Laplace equation*

$$\Delta f = 0.$$

Proposition 5.2. *The real and the imaginary part of a holomorphic function are harmonic.*

Proof. **Exercise.** □

Proposition 5.3. *Let $U, V \subseteq \mathbb{C}$ be open and $f \in \mathcal{O}(U)$ such that $V \subseteq f(U)$. If $g : V \to \mathbb{R}$ is harmonic, then $g \circ f : U \to \mathbb{R}$ is also harmonic.*

Proof. **Exercise.** □

Lemma 5.4. *Let $D = \mathbb{C}$ or $D = \mathbb{D}$ and $u : D \to \mathbb{R}$ a harmonic function. Then, there exists a harmonic function $v : D \to \mathbb{R}$ such that $u + \mathrm{i}v \in \mathcal{O}(D)$.*

Proof. Define the continuously partially differentiable function $v : D \to \mathbb{R}$ given by

$$v(x, y) := \int_0^y u_x(x, t)\,\mathrm{d}t - \int_0^x u_y(s, 0)\,\mathrm{d}s \quad \forall (x, y) \in D.$$

Differentiating by $\partial/\partial x$ and using that u is harmonic we get

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\,\mathrm{d}t - u_y(x, 0) \\ &= -\int_0^y u_{yy}(x, t)\,\mathrm{d}t - u_y(x, 0) \\ &= -u_y(x, y) + u_y(x, 0) - u_y(x, 0) \\ &= -u_y(x, y). \end{aligned}$$

Note that the interchange of differentiation and integration in the first step is permitted since the integrand is continuously differentiable and the integration range compact. On the other hand, differentiating by $\partial/\partial y$ we obtain

$$v_y(x,y) = u_x(x,y).$$

Thus, the pair (u,v) satisfies the Cauchy-Riemann equations so that $u + \mathrm{i}v$ is holomorphic according to Proposition 1.3. □

Theorem 5.5. *Let $D \subseteq \mathbb{C}$ be a homologically simply connected region and $u : D \to \mathbb{R}$ be harmonic. Then, there exists a harmonic function $v : D \to \mathbb{R}$ such that $u + \mathrm{i}v : D \to \mathbb{C}$ is holomorphic.*

Proof. If $D = \mathbb{C}$ then Lemma 5.4 directly applies and we are done. Suppose therefore that $D \neq \mathbb{C}$. By the Riemann Mapping Theorem 4.36 there exists a biholomorphic map $f : D \to \mathbb{D}$. By Proposition 5.3, $u \circ f^{-1} : \mathbb{D} \to \mathbb{R}$ is harmonic. Applying Lemma 5.4, there exists a harmonic function $w : \mathbb{D} \to \mathbb{C}$ such that $u \circ f^{-1} + \mathrm{i}w : \mathbb{D} \to \mathbb{C}$ is holomorphic. Define $v : D \to \mathbb{R}$ by $v := w \circ f$. Then, v is harmonic by Proposition 5.3 and $u + \mathrm{i}v : D \to \mathbb{C}$ is holomorphic. □

Proposition 5.6. *Harmonic functions are infinitely differentiable.*

Proof. **Exercise.** □

Theorem 5.7 (Mean Value Theorem)**.** *Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ harmonic. Suppose $a \in D$ and $r > 0$ such that $\overline{B_r(a)} \subset D$. Then,*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

Proof. Choose $s > r$ such that $B_s(a) \subseteq D$. By Theorem 5.5 there exist a harmonic function $v : B_s(a) \to \mathbb{C}$ such that $f := u + \mathrm{i}v : B_s(a) \to \mathbb{C}$ is holomorphic. Applying the Cauchy Integral Formula (Theorem 2.20) to f at the point a with path $\partial B_r(a)$ we obtain,

$$f(a) = \frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(a)} \frac{f(\zeta)}{\zeta - a} \mathrm{d}\zeta = \frac{1}{2\pi} \int_0^{2\pi} f\left(a + re^{\mathrm{i}\theta}\right) \mathrm{d}\theta.$$

Taking the real part on both sides yields the desired result. □

Definition 5.8. Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ continuous. We say that u has the *mean value property* iff for all $a \in D$ and all $r > 0$ such that $\overline{B_r(a)} \subset D$ we have

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u\left(a + re^{\mathrm{i}\theta}\right) \, \mathrm{d}\theta.$$

It turns out that the mean value property implies harmonicity.

Theorem 5.9 (Maximum Principle)**.** *Let $D \subseteq \mathbb{C}$ be a region and $u : D \to \mathbb{R}$ a continuous function with the mean value property. Suppose that u has a maximum at some point $a \in D$, i.e., that $u(z) \leq u(a)$ for all $z \in D$. Then u is constant.*

Proof. Define

$$A := \{z \in D : u(z) = u(a)\}.$$

Since u is continuous, A must be closed in D. We proceed to show that A is also open. Let $z_0 \in A$ and $r > 0$ such that $B_r(z_0) \subset D$. Choose $b \in B_r(z_0)$ and set $s := |b - z_0|$. By the mean value property

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(z_0 + se^{\mathrm{i}\theta}\right) \, \mathrm{d}\theta.$$

The integrand is continuous and everywhere smaller or equal to $u(z_0)$. Hence, for the equality to hold, we must have $u\left(z_0 + se^{\mathrm{i}\theta}\right) = u(z_0)$ for all $\theta \in [0, 2\pi)$. In particular, $u(b) = u(z_0) = u(a)$ and hence $b \in A$. Since b was chosen arbitrarily we have $B_r(z_0) \subseteq A$, showing that A is open. Since A is non-empty, closed in D and open, we must have $A = D$. Thus, $u(z) = u(a)$ for all $z \in D$ and u is constant. □

Proposition 5.10. *Let $D \subset \mathbb{C}$ be a bounded region and $u : \overline{D} \to \mathbb{R}$ a continuous function with the mean value property in D, satisfying $u|_{\partial D} = 0$. Then, $u = 0$.*

Proof. **Exercise.** □

Exercise 61. Show the following version of the maximum principle, which is more similar to Theorem 2.33: Let $D \subseteq \mathbb{C}$ be a region and $f : D \to \mathbb{C}$ a continuous function satisfying the mean value property. Suppose that $|f|$ has a maximum at some point $a \in D$, i.e., that $|f(z)| \leq |f(a)|$ for all $z \in D$. Then f is constant. [Hint: Consider the function $g(z) := \Re(f(z)/f(a))$.]

5.2 The Dirichlet Problem

Definition 5.11. The function $P : \mathbb{D} \to \mathbb{R}$ given by

$$P(z) := \Re\left(\frac{1+z}{1-z}\right) \quad \forall z \in \mathbb{D}$$

is called the *Poisson kernel.* For $0 \le r < 1$ and $\theta \in \mathbb{R}$ it is also common to use the notation

$$P_r(\theta) := P\left(r e^{\mathrm{i}\theta}\right).$$

Proposition 5.12. *The Poisson kernel P has the following properties:*

1. *P is harmonic.*
2. *For all $z = re^{\mathrm{i}\theta} \in \mathbb{D}$,*

$$P_r(\theta) = P(z) = \frac{1-|z|^2}{|1-z|^2} = \frac{1-r^2}{1-2r\cos\theta + r^2}.$$

3. *$P(z) > 0$ for all $z \in \mathbb{D}$.*
4. *$P_r(-\theta) = P(\overline{z}) = P(z) = P_r(\theta)$ for all $z = re^{\mathrm{i}\theta} \in \mathbb{D}$.*
5. *for all $z = re^{\mathrm{i}\theta} \in \mathbb{D}$,*

$$P_r(\theta) = P(z) = 1 + \sum_{n=1}^{\infty} (z^n + \overline{z}^n) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{\mathrm{i}n\theta}.$$

6. *For all $0 \le r < 1$ we have*

$$\frac{1}{2\pi\mathrm{i}} \int_{\partial B_r(0)} \frac{P(\zeta)}{\zeta}\,\mathrm{d}\zeta = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,\mathrm{d}\theta = 1.$$

7. *For all $0 < r < 1$ and $0 < |\delta| < |\theta| \le \pi$ we have $P_r(\theta) < P_r(\delta)$.*
8. *For each $0 < \delta < \pi$ and $\epsilon > 0$ there exists $0 < \rho < 1$ such that for all $\rho < r < 1$ and $\delta < |\theta| \le \pi$ we have $|P_r(\theta)| < \epsilon$.*

Proof. 1. By definition, the Poisson kernel is the real part of a holomorphic function. Thus, it is harmonic by 5.2. 2. Elementary calculation. 3. This follows immediately from 2. 4. This follows immediately from 2. 5. **Exercise.**

6. Note that for $0 \leq r < 1$ the series representation given in 5. converges uniformly. So, we can exchange summation and integration to get,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\,\mathrm{d}\theta = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\mathrm{i}n\theta}\,\mathrm{d}\theta = 1.$$

7. This follows easily from 2. 8. Fix $0 < \delta < \pi$ and $\epsilon > 0$. Then, $P_r(\delta) \to 0$ for $r \to 1-$ using 2. Thus, there is $0 < \rho < 1$ so that $|P_r(\delta)| < \epsilon$ if $\rho < r < 1$. Using 7. completes the proof of 8. □

Theorem 5.13. *Let $b : \partial\mathbb{D} \to \mathbb{R}$ be continuous. Then, there exists a unique continuous function $u : \overline{\mathbb{D}} \to \mathbb{R}$ such that $u|_{\partial\mathbb{D}} = b$ and u is harmonic in $\mathbb{D}$. Moreover, for all $0 \leq r < 1$ and $\theta \in \mathbb{R}$,*

$$u\left(re^{\mathrm{i}\theta}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi) b\left(e^{\mathrm{i}\phi}\right)\,\mathrm{d}\phi.$$

Proof. Define $u(z)$ for $z \in \mathbb{D}$ by the stated formula and $u(z) := b(z)$ for $z \in \partial\mathbb{D}$. We first show that u is harmonic in $\mathbb{D}$. We note that for $z \in \mathbb{D}$,

$$\begin{aligned} u(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Re\left(\frac{1+ze^{-\mathrm{i}\phi}}{1-ze^{-\mathrm{i}\phi}}\right) b\left(e^{\mathrm{i}\phi}\right)\,\mathrm{d}\phi \\ &= \Re\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{\mathrm{i}\phi}+z}{e^{\mathrm{i}\phi}-z} b\left(e^{\mathrm{i}\phi}\right)\,\mathrm{d}\phi\right). \end{aligned}$$

Note that the integrand in the last expression is continuous as a function of $(\phi, z) \in \mathbb{R} \times \mathbb{D}$ and holomorphic as a function of $z \in \mathbb{D}$ for each value of $\phi \in \mathbb{R}$. That is, we can apply Lemma 2.42 to conclude that the integral defines a holomorphic function in $\mathbb{D}$. But by Proposition 5.2, the real part of this function is harmonic.

We proceed to show that u is continuous in $\overline{\mathbb{D}}$. Since continuity in $\mathbb{D}$ follows from harmonicity it suffices to consider points in the boundary of $\overline{\mathbb{D}}$. In particular, it is enough to show the following: Given $\psi \in [-\pi, \pi)$ and $\epsilon > 0$, there exist $\delta > 0$ and $0 < \rho < 1$ such that

$$\left|u\left(re^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right)\right| < \epsilon \quad \forall \rho < r \leq 1, \forall\theta \in (\psi-\delta, \psi+\delta).$$

We proceed to find such δ and ρ given ψ and ϵ. By continuity of b, there exists $\delta > 0$ such that

$$\left|b\left(e^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right)\right| < \frac{\epsilon}{2} \quad \forall\theta \in (\psi-\delta, \psi+\delta).$$

By Proposition 5.12.8, there exists $0 < \rho < 1$ such that

$$P_r(\theta) < \frac{\epsilon}{4(\|b\|_{\partial\mathbb{D}} + 1)} \quad \forall \rho < r < 1, \forall \delta/2 < |\theta| \leq \pi.$$

Now let $\theta \in (\psi - \delta, \psi + \delta)$ and $0 < r < \rho$. Then,

$$\begin{aligned}
\left|u\left(re^{\mathrm{i}\theta}\right) - b\left(e^{\mathrm{i}\psi}\right)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi) b\left(e^{\mathrm{i}\phi}\right)\,\mathrm{d}\phi - b\left(e^{\mathrm{i}\psi}\right)\right| \\
&= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)\left(b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right)\right)\,\mathrm{d}\phi\right| \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)\left|b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right)\right|\,\mathrm{d}\phi \\
&= \frac{1}{2\pi}\int_{|\phi-\psi|<\delta} P_r(\theta-\phi)\left|b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right)\right|\,\mathrm{d}\phi \\
&\quad + \frac{1}{2\pi}\int_{|\phi-\psi|\geq\delta} P_r(\theta-\phi)\left|b\left(e^{\mathrm{i}\phi}\right) - b\left(e^{\mathrm{i}\psi}\right)\right|\,\mathrm{d}\phi \\
&\leq \frac{1}{2\pi}\int_{|\phi-\psi|<\delta} P_r(\theta-\phi)\frac{\epsilon}{2}\,\mathrm{d}\phi \\
&\quad + \frac{1}{2\pi}\int_{|\phi-\psi|\geq\delta} \frac{\epsilon}{4(\|b\|_{\partial\mathbb{D}}+1)} 2\|b\|_{\partial\mathbb{D}}\,\mathrm{d}\phi \\
&< \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\phi)\frac{\epsilon}{2}\,\mathrm{d}\phi + \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\epsilon}{2}\,\mathrm{d}\phi \\
&= \epsilon.
\end{aligned}$$

Here, we have used the properties of the Poisson kernel given in Proposition 5.12 parts 3. and 6.

It remains to show uniqueness of the function u. Suppose there was another function $v : \overline{\mathbb{D}} \to \mathbb{R}$ with the required properties. Then, the difference $u - v$ would be continuous on $\overline{\mathbb{D}}$ and harmonic in $\mathbb{D}$. Furthermore, $(u - v)|_{\partial\mathbb{D}} = 0$, so by Proposition 5.10, $u - v = 0$, i.e., $u = v$. □

Definition 5.14. We call a region $D \subseteq \mathbb{C}$ *disk-like* iff there exists a conformal equivalence $D \to \mathbb{D}$ which extends to a homeomorphism $\overline{D} \to \overline{\mathbb{D}}$.

Remark 5.15. A disk-like region is in particular homologically simply connected and bounded.

Theorem 5.16. *Let $D \subset \mathbb{C}$ be a disk-like region. Let $b : \partial D \to \mathbb{R}$ be continuous. Then, there exists a unique continuous function $u : \overline{D} \to \mathbb{R}$ such that $u|_{\partial\mathbb{D}} = b$ and u is harmonic in D.*

Proof. **<u>Exercise</u>.** □

Theorem 5.17. *Let $U \subseteq \mathbb{C}$ be open and $u : U \to \mathbb{R}$ be continuous with the mean value property. Then, u is harmonic.*

Proof. Let $a \in U$ and $r > 0$ such that $\overline{B_r(a)} \subset U$. It is sufficient to show that u is harmonic in $B_r(a)$. Since $B_r(a)$ is disk-like there exists by Theorem 5.16 a continuous function $v : \overline{B_r(a)} \to \mathbb{R}$ which is harmonic in $B_r(a)$ and coincides with u in $\partial B_r(a)$. But the difference $u - v : \overline{B_r(a)} \to \mathbb{R}$ is continuous, has the mean value property in $B_r(a)$ and vanishes on the boundary $\partial B_r(a)$. Thus $u = v$ also in $B_r(a)$ by Proposition 5.10. In particular, u is harmonic in $B_r(a)$. □